1974

Algorithms for Kalman filters with delayed state measurements

Vernon Leon Schwenk
Iowa State University

74-23,764

SCHWENK, Vernon Leon, 1938-
ALGORITHMS FOR KALMAN FILTERS WITH DELAYED STATE MEASUREMENTS.

Iowa State University, Ph.D., 1974
Engineering, electrical

University Microfilms, A XEROX Company, Ann Arbor, Michigan

Algorithms for Kalman filters with delayed state measurements

by

Vernon Leon Schwenk

A Dissertation Submitted to the
Graduate Faculty in Partial Fulfillment of
The Requirements for the Degree of
DOCTOR OF PHILOSOPHY

Major: Electrical Engineering

Approved:

Signature was redacted for privacy.

In'Charge of Major Work

Signature was redacted for privacy.

For the Major Department

Signature was redacted for privacy.

For the Graduate College

Iowa State University
Ames, Iowa

1974

TABLE OF CONTENTS

CHAPTER I. THE DISCRETE KALMAN FILTER

Introduction

An important class of problems in communications and control consists of finding an estimate of some quantity, given measurements which are composed of the unknown quantity plus additive noise. One of the first studies of this kind was by Gauss in 1809 when determining orbital parameters of a celestial body (1). In modern times one of the most significant results was obtained by Wiener (2) in 1949 who gave an integral equation solution to the problem of estimating random signals. However, this integral equation could be solved explicitly only for certain special cases. Until about 1960 almost all contributions to the area were obtained essentially from Wiener's original work and subsequent extensions. By this time researchers had begun to realize the value of the modern digital computer and numerical solutions began to assume a more significant role than analytic solutions.

One particular recursive procedure generally is considered to have stimulated great interest in the area of estimation. This was suggested by Kalman (3) in 1960 who formulated the problem using the concepts of state and state transition in the representation of the random signals. In the state formulation for the discrete-time case, linear systems are specified by simultaneous first-order difference equations. The discrete-time case seems to be inherently suited for solution on a digital computer and accurately describes the common physical situation in which measurements are obtained at discrete instants of time (4). From a computational viewpoint one of the most significant aspects of the Kalman approach is that the estimates are obtained recursively as new measurements are made. For historical reasons, methods of obtaining estimates of unknown quantities generally are referred to as filters.

The filter described by Kalman is a means of obtaining an estimate of the state of a linear, discrete-time system. The estimate is a linear combination of noise corrupted observations taken at discrete instances of time of a linear function of the state of the system at the same values of time. In the case of Gaussian noise sequences the linear estimate is optimal for a wide class of loss functions including the quadratic case. This important result is stated by Nahi (5) and is attributed to Doob by Meditch (6). Since the original paper there has been a number of extensions by various authors. Several of these that are of particular value are described in this chapter, together with a statement of the discrete-time filtering problem and the solution by Kalman.

In 1968, Kalman's solution was extended by Brown and

Hartman (7) to include the case in which the measurement is a linear function of both the system state at the time of the measurement and of the state at the preceding instant of time.

The Brown and Hartman filter while applicable for a more general measurement case does not provide estimates for as large a class of conditions or in as convenient a manner as desired. For example contributions to the field of conventional Kalman filtering allow the effects of simultaneous measurements to be examined individually and permit the use of some suboptimal techniques designed to reduce computational effort. It is the purpose of this dissertation to present for the more general measurement case algorithms that provide estimates for these and other conditions that have been found in practice to be of substantial significance.

The Conventional Kalman Filter

The discrete filtering problem solved by Kalman (3) is as follows:

Problem 1: Consider the system

$$\text{process:} \quad x(n+1) = \phi(n)x(n) + B(n)u(n) \tag{1.1}$$

$$\text{measurement:} \quad y(n) = M(n)x(n) + v(n) \tag{1.2}$$

where the state x is an n vector, the forcing function u is an r vector, the measurement or observation y is an m vector, the measurement noise or error v is an m vector and n is the discrete time index. Also, ϕ is the nxn state transition matrix, B is the nxr input matrix and M is the mxn measurement matrix. The stochastic process $\{u(n),\ n = 0,1,\ldots\}$ is a zero mean Gaussian sequence with covariance

$$\mathrm{cov}[u(n)] = E[u(n)u'(m)] = Q(n)\delta_{mn} \tag{1.3}$$

where δ is the Kronecker delta and the prime mark denotes matrix transpose. The process $\{v(n)\}$ is a zero mean Gaussian sequence with covariance

$$\mathrm{cov}[v(n)] = R(n)\delta_{mn} \tag{1.4}$$

The two processes usually are considered to be uncorrelated so

$$E[u(n)v'(m)] = 0\ . \tag{1.5}$$

The initial state x(0) is a zero-mean Gaussian random vector with covariance

$$\mathrm{cov}[x(0)] = P(0) \tag{1.6}$$

The initial state is assumed to be uncorrelated with $\{u(n)\}$ and $\{v(n)\}$ so

$$E[x(0)u'(n)] = 0 \tag{1.7}$$

$$E[x(0)v'(n)] = 0 \ . \tag{1.8}$$

Given $P(0)$ and the set of measurements $\{y(1),y(2),...,y(n)\}$ the problem is to find the optimal estimate of $x(n)$, $\hat{x}(n|n)$, which minimizes the mean square filtering error.

The filtering error is the difference between the actual state and the estimate so the function to be minimized is

$$J = E\{[x(n)-\hat{x}(n|n)]'[x(n)-\hat{x}(n|n)]\} \tag{1.9}$$

It may seem that the Gaussian assumption for $\{u\}$ and $\{v\}$ is substantially restrictive, but this is not the case since many physical processes are approximately Gaussian (8). In many cases when a large number of small independent random effects are superimposed the distribution of the sum of the effects is approximately Gaussian under certain general conditions. This is actually an approximate statement of the central limit theorem of probability theory. See, for example, Parzen (9) for a treatment of this concept.

The solution to Problem 1 has been obtained in recursive form by Kalman (3) and is stated as follows:

The solution to Problem 1 is the optimal estimate given by

$$\hat{x}(n|n) = \hat{x}(n|n-1) + K(n)[y(n)-M(n)\hat{x}(n|n-1)] \tag{1.10}$$

$$\hat{x}(n|n-1) = \phi(n-1)\hat{x}(n-1|n-1) \tag{1.11}$$

$$K(n) = P(n|n-1)M'(n)[M(n)P(n|n-1)M'(n)+R(n)]^{-1} \tag{1.12}$$

$$P(n|n-1) = \phi(n-1)P(n-1|n-1)\phi'(n-1) + B(n-1)Q(n-1)B'(n-1) \tag{1.13}$$

$$P(n|n) = P(n|n-1)-K(n)M(n)P(n|n-1) \tag{1.14}$$

with initial conditions

$$\hat{x}(0|0) = 0 \tag{1.15}$$

$$P(0|0) = P(0) \tag{1.16}$$

The nxn matrix $P(n|n)$ is the covariance of the filtering error $\hat{x}(n)-x(n|n)$. The initial error covariance, $P(0|0)$, is equal to the covariance of the initial state, $P(0)$, since the initial estimate, $\hat{x}(0|0)$, is zero for the case of zero-mean initial state.

The nxm matrix $K(n)$ generally is called the gain matrix or simply the gain of the filter.

The conditional notation $(l|m)$ indicates the value of a quantity at time l given measurement data through time m. Thus $\hat{x}(n|n-1)$ is the optimal estimate of the state at time n given measurements of the state through time n-1. The co-variance $P(n|n-1)$ is

$$P(n|n-1) = \mathrm{cov}[x(n)-\hat{x}(n|n-1)] \tag{1.17}$$

Calculation of the quantities $\hat{x}(n|n-1)$ and $P(n|n-1)$ does not depend on the measurement at time n and is referred to commonly as the time update at time n. The remainder of the quantities in the algorithm constitutes the measurement update since it is a function of the measurement at time n.

There are several good tutorial references available which develop the Kalman filter including Meditch (6) and Nahi (5).

Sequential Measurement Processing

In the conventional Kalman filter all m measurements available at a given time are processed simultaneously. As shown by Sorenson (4) it is possible to process the data sequentially if the measurement vector can be partitioned into components with uncorrelated measurement errors. In sequential processing each set of data is treated separately. This technique is summarized as follows:

Suppose that at a given time n the m dimensional measurement vector can be partitioned into p components as

$$y(n) = \begin{bmatrix} y_1(n) \\ y_2(n) \\ \vdots \\ y_p(n) \end{bmatrix} = \begin{bmatrix} M_1(n) \\ M_2(n) \\ \vdots \\ M_p(n) \end{bmatrix} x(n) + \begin{bmatrix} v_1(n) \\ v_2(n) \\ \vdots \\ v_p(n) \end{bmatrix} \tag{1.18}$$

where

$$E\{v_i(n)v_j'(n)\} = R_i(n)\delta_{ij} \tag{1.19}$$

Then the solution to Problem 1 is the estimate computed by using the Kalman algorithm to obtain $\hat{x}_1(n|n)$ and $P_1(n|n)$ from $y_1(n)$ and the following algorithm p-1 times to process $y_2(n)\ldots y_p(n)$

$$K_i(n) = P_{i-1}(n|n)M_i'(n)[M_i(n)P_{i-1}(n|n)M_i'(n) + R_i(n)]^{-1} \tag{1.20}$$

$$P_i(n|n) = P_{i-1}(n|n) - K_i(n)M_i(n)P_{i-1}(n|n) \tag{1.21}$$

$$\hat{x}_i(n|n) = \hat{x}_{i-1}(n|n) + K_i(n)[y_i(n)-M_i(n)\hat{x}_{i-1}(n|n)] \tag{1.22}$$

where

$$i = 2,3,\ldots,p \tag{1.23}$$

The optimal estimate $\hat{x}(n|n)$ and covariance $P(n|n)$ are $\hat{x}_p(n|n)$ and $P_p(n|n)$.

Error Covariance for Suboptimal Gain

The covariance calculation $P(n|n)$ given by the Kalman algorithm is an optimal calculation in that the expression is valid only if the gain matrix is as given in the algorithm. In those cases in which some suboptimal technique is used to obtain a gain matrix, a general expression for covariance is required. Such a result has been obtained by several authors including Meditch (6) and is stated as follows:

Consider Problem 1. In the event that a suboptimal estimate is to be obtained by using an arbitrary gain matrix, the covariance of the resulting estimation error $P(n|n)$ is given as.

$$P(n|n) = [I-K(n)M(n)]P(n|n-1)[I-K(n)M(n)]' + K(n)R(n)K'(n) \tag{1.24}$$

where I is the nxn identity matrix.

Alternate Form of the Kalman Filter

The optimal filter also may be implemented in a form in which the inverse of the estimation error covariance matrix is propagated. This technique is attributed by Kaminski et al. (10) to D. C. Fraser and is stated as follows:

The optimal estimate for Problem 1 may be obtained from $d(n|n)$, $d(n|n-1)$, $P^{-1}(n|n)$ and $P^{-1}(n|n-1)$ using the definitions

$$d(n|n) = P^{-1}(n|n)\hat{x}(n|n) \tag{1.25}$$

$$d(n|n-1) = P^{-1}(n|n-1)x(n|n-1) \tag{1.26}$$

and

$$d(n|n-1) = [I-L(n)B'(n-1)]\phi^{-T}(n-1)d(n-1\ n-1) \tag{1.27}$$

$$P^{-1}(n|n-1) = [I-L(n)B'(n-1)]F(n) \tag{1.28}$$

$$F(n) = \phi^{-T}(n-1)P^{-1}(n-1|n-1)\phi^{-1}(n-1) \tag{1.29}$$

$$L(n) = F(n)B(n-1)[Q^{-1}(n-1)+B'(n-1)F(n)B(n-1)]^{-1} \tag{1.30}$$

$$d(n|n) = d(n|n-1) + M'(n)R^{-1}(n)y(n) \tag{1.31}$$

$$P^{-1}(n|n) = P^{-1}(n|n-1) + M'(n)R^{-1}(n)M(n) \tag{1.32}$$

where

$$\phi^{-T} = (\phi^{-1})' \tag{1.33}$$

CHAPTER II. KALMAN FILTERS WITH DELAYED STATE MEASUREMENTS

Doppler Counts as Measurements

As mentioned previously Kalman's solution was extended to include the case in which the measurement at a given time is dependent on the state at that time as well as the previous discrete time. This particular form of measurement was noted by Brown and Hagerman (11) in 1968 with reference to frequency counts as measurements. One example of measurement data of this type is that from the Navy Navigation Satellite System or TRANSIT described by Stansell (12).

For the TRANSIT system the measurement is a count of the number of difference frequency cycles between a reference and a signal containing a doppler shift due to motion of the transmitter. This type of measurement is called an integrated doppler measurement by Stansell (12) because the frequency count is represented mathematically by an integral of the difference frequency over an interval of time. Thus if ρ is the position separation or range between transmitter and receiver, f_D the doppler frequency and c the velocity of propagation, the doppler count N(n) over the time period t=n-1 to t=n is

$$N(n) = \int_{n-1}^{n} (f_r - f_0)\,dt \tag{2.1}$$

$$= \int_{n-1}^{n} f_D \; dt \tag{2.2}$$

$$= -\int_{n-1}^{n} \left(\frac{f_0}{c}\right) \dot{\rho} \; dt \tag{2.3}$$

$$= - \frac{f_0}{c}[\rho(n) - \rho(n-1)] \tag{2.4}$$

where f_0 is the transmitted frequency and f_r is the received frequency. Thus, for example, if the range ρ or one of its components is one of the describing state variables of the system, the measurement equation would be of the form

$$y(n) = M(n)x(n) + N(n)x(n-1) + v(n) \tag{2.5}$$

where $v(n)$ is any uncorrelated measurement noise.

In the Kalman format the optimal estimation problem for a system with a measurement of this type may be stated as follows:

Problem 2: Consider the system

process: $x(n+1) = \phi(n)x(n) + B(n)u(n)$ (2.6)

measurement: $y(n) = M(n)x(n) + N(n)x(n-1) + v(n)$ (2.7) where N is a mxn measurement matrix and the other matrices and vectors are as defined previously with

$$\mathrm{cov}[u(n)] = Q(n)\delta_{mn} \tag{2.8}$$

$$\mathrm{cov}[v(n)] = R(n)\delta_{mn} \tag{2.9}$$

$$E[u(n)v'(m)] = 0 \tag{2.10}$$

$$\mathrm{cov}[x(0)] = P(0) \tag{2.11}$$

$$E[x(0)u'(n)] = 0 \tag{2.12}$$

$$E[x(0)v'(n)] = 0 \tag{2.13}$$

Given P(0) and the set of measurements $\{y(1), y(2), \ldots, y(n)\}$ the problem is to find the optimal estimate of x(n), $\hat{x}(n|n)$, which minimizes the mean square filtering error.

The Brown-Hartman Algorithm

The solution to Problem 2 as stated by Brown and Hartman (7) is given below.

The solution to Problem 2 is the optimal estimate given by

$$\hat{x}(n|n) = \hat{x}(n|n-1) + K(n)[y(n)-M(n)\hat{x}(n|n-1) - N(n)\hat{x}(n-1|n-1)] \tag{2.14}$$

$$\hat{x}(n|n-1) = \phi(n-1)\hat{x}(n-1|n-1) \tag{2.15}$$

$$\begin{aligned} S(n) = {} & M(n)P(n|n-1)M'(n) + N(n)P(n-1|n-1)N'(n) \\ & + N(n)P(n-1|n-1)\phi'(n-1)M'(n) + R(n) \\ & + M(n)\phi(n-1)P(n-1|n-1)N'(n) \end{aligned} \tag{2.16}$$

$$K(n) = [P(n|n-1)M'(n) + \phi(n-1)P(n-1|n-1)N'(n)]S^{-1}(n) \tag{2.17}$$

$$\begin{aligned} P(n|n-1) = {} & \phi(n-1)P(n-1|n-1)\phi'(n-1) \\ & + B(n-1)Q(n-1)B'(n-1) \end{aligned} \tag{2.18}$$

$$\begin{aligned} P(n|n) = {} & P(n|n-1) - K(n)[M(n)P(n|n-1) \\ & + N(n)P(n-1|n-1)\phi'(n-1)] \end{aligned} \tag{2.19}$$

with initial conditions

$$\begin{aligned} \hat{x}(0|0) &= 0 \\ P(0|0) &= P(0) \end{aligned} \tag{2.20}$$

The nxn matrix P(n n) is the covariance of the filtering error $x(n)-\hat{x}(n|n)$. The initial error covariance is equal to the covariance of the initial state since the initial estimate is zero for the zero-mean initial state case.

It is common practice to refer to any filtering algorithm that propagates the estimation error covariance as a covariance filter.

Some important properties of the system described in Problem 2 and its optimal estimate that will be of value in later chapters are stated below.

Comment 1: $E[x(n)v'(m)] = 0 \quad \text{all } m,n$ (2.21)

To obtain this result start with (2.6) and observe that

$$x(n+1) = [\prod_{i=1}^{n} \Phi(i)]x(0) + \sum_{i=0}^{n} [\prod_{j=i+1}^{n} \phi(j)]B(i)u(i) \qquad (2.22)$$

where the products defined by Π are taken on the left and

$$\prod_{k}^{k-1} \phi(j) = I \qquad (2.23)$$

the nxn identity matrix.

Then

$$E[x(n)v'(m)] = E\{[\prod_{i=0}^{n-1} \phi(i)]x(0)v'(m)\} + E\{\sum_{i=0}^{n-1} [\prod_{j=i+1}^{n-1} \phi(j)]B(i)u(i)v'(m)\} \qquad (2.24)$$

$$= [\prod_{i=0}^{n-1} \phi(i)]E[x(0)v'(m)]$$

$$+\sum_{i=0}^{n-1}\left[\prod_{j=i+1}^{n-1}\phi(j)\right]B(i)E[u(i)v'(m)] \tag{2.25}$$

$$= 0 \tag{2.26}$$

using (2.13) and (2.10).

<u>Comment 2</u>: $E[x(n)u'(m)] = 0 \quad m \geq n$ (2.27)

To obtain this result, proceed similarly to Comment 1 and

$$E[x(n)u'(m)] = \left[\prod_{i=0}^{n-1}\phi(i)\right]E[x(0)u'(m)]$$

$$+\sum_{i=0}^{n-1}\left[\prod_{j=i+1}^{n-1}\phi(i)\right]B(i)E(u(i)u'(m)] \tag{2.28}$$

$$=0 \tag{2.29}$$

using (2.12) and the result from (2.8)

$$E[u(i)u'(m)] = 0 \quad i \neq m \tag{2.30}$$

and

$$i = 0,1,\ldots,n-1 \tag{2.31}$$

<u>Comment 3</u>: $E[\hat{x}(j|j)v'(k)] = 0 \quad k>j$ (2.32)

To obtain this result start with (2.14) and (2.15) to show that $\hat{x}(j|j)$ is a linear combination of measurements as

$$\hat{x}(n|n) = \sum_{i=1}^{n}\left[\prod_{j=i+1}^{n}A(j)\right]K(i)y(i) \tag{2.33}$$

The products defined by Π are taken on the left and

$$A(j) = \phi(j-1) - K(j)M(j)\phi(j-1) - K(j)N(j) \tag{2.34}$$

and

$$\prod_{n+1}^{n} A(j) = I \tag{2.35}$$

where I is the nxn identity matrix.

Then using (2.7)

$$E[\hat{x}(n|n)v'(k)] = \sum_{i=1}^{n} [\prod_{j=i+1}^{n} A(j)]K(i)E[M(i)x(i)$$

$$+ N(i)x(i-1)+v(i)]v'(k) \tag{2.36}$$

$$= \sum_{i=1}^{n} [\prod_{j=i+1}^{n} A(j)]K(i)[M(i)Ex(i)v'(k)$$

$$+ N(i)Ex(i-1)v'(k) + Ev(i)v'(k)] \tag{2.37}$$

$$= 0 \quad k>n \tag{2.38}$$

using Comment 1 and (2.9).

<u>Comment 4</u>: $E[\hat{x}(j|j)u'(k)] = 0 \quad k\geq j$ (2.39)

To obtain this result, proceed similarly to Comment 3 so

$$E[\hat{x}(n|n)u'(k)] = \sum_{i=1}^{n} [\prod_{j=i+1}^{n} A(j)]K(i)Ey(i)u'(k) \quad (2.40)$$

$$= \sum_{i=1}^{n} [\prod_{j=i+1}^{n} A(j)]K(i)[M(i)Ex(i)u'(k)$$

$$+ N(i)Ex(i-1)u'(k) + Ev(i)u'(k)] \quad (2.41)$$

$$= 0 \quad k \geq n \quad (2.42)$$

using Comment 2 and (2.10).

Numerical Properties of the Filter

Some numerical properties of the Brown-Hartman covariance filter have been studied by Stuva (13) and compared with another form of measurement update suggested by Brown. This new update is

$$S(n) = [M(n)\phi(n-1)+N(n)]P(n-1|n-1)[M(n)\phi(n-1)$$

$$+ N(n)]' + M(n)B(n-1)Q(n-1)B'(n-1)M'(n) + R(n) \quad (2.43)$$

$$K(n) = \{\phi(n-1)P(n-1|n-1)[M(n)\phi(n-1)+N(n)]'$$

$$+ B(n-1)Q(n-1)B'(n-1)M'(n)\}S^{-1}(n) \quad (2.44)$$

$$P(n|n) = P(n|n-1) - K(n)S(n)K'(n) \quad (2.45)$$

$$\hat{x}(n|n) = \hat{x}(n|n-1) + K(n)\{y(n) - [M(n)\phi(n-1) + N(n)]\hat{x}(n-1|n-1)\} \tag{2.46}$$

The study involved investigation of the numerical accuracy of the calculation of S(n) and of position error components of K(n) for an integrated inertial/doppler-satellite navigation system. The problem arises from the fact that errors in fixed word length digital machines tend to accumulate when a large number of iterations is made. The parameters that were varied in the study were the time increment Δt between measurements and the measurement noise. In all cases the new algorithm gave more accurate results, particularly for the cases of small Δt or small measurement noise.

CHAPTER III. SEQUENTIAL PROCESSING OF DELAYED MEASUREMENTS

A method of processing the scalar components of a vector measurement individually or in small groups is extremely valuable since it allows the effects of each to be examined independently. This sequential processing technique also has significant computational advantages since matrix inverse operations are reduced to the trivial scalar case or at least to smaller size matrices. Inverses of large matrices are undesirable because of the requirements for time and storage and the accuracy of the end result.

The sequential processing technique for the conventional Kalman filter cannot be applied for the delayed state measurement case because of the dependence of the measurement on the previous state. Whenever a component of the measurement is processed, new information about the previous state is introduced which must be utilized in an optimal manner. The new results presented here utilize the information contained in the components of the delayed state measurement optimally for any partitions of the measurement vector for which the measurement errors are uncorrelated.

The sequential technique may be of some value for the case in which time intervals for various measurement components are not exactly coincidental. In certain cases even small misalignment can lead to gross inaccuracies if the estimate calculation is sensitive to the measurement matrices (14). With sequential processing the measurements can be handled individually with the measurement matrices for each evaluated at the appropriate instants of time.

The delayed state sequential processing algorithm is as follows:

If the measurement vector can be partitioned into P components

$$y(n) = \begin{bmatrix} y_1(n) \\ y_2(n) \\ \vdots \\ y_p(n) \end{bmatrix} = \begin{bmatrix} M_1(n) \\ M_2(n) \\ \vdots \\ M_p(n) \end{bmatrix} x(n)$$

$$+ \begin{bmatrix} N_1(n) \\ N_2(n) \\ \vdots \\ N_p(n) \end{bmatrix} x(n-1) + \begin{bmatrix} v_1(n) \\ v_2(n) \\ \vdots \\ v_p(n) \end{bmatrix} \tag{3.1}$$

such that

$$E\{v_i(n)v_j'(n)\} = \delta_{ij}R_i(n) \tag{3.2}$$

the optimal estimate can be obtained by using the delayed state covariance filter to obtain $\hat{x}_1(n|n)$ and $P_1(n|n)$ from $y_1(n)$ and the following algorithm p-1 times to process

$y_2(n)\ldots y_p(n)$ with the index i assuming the values $2,\ldots,p$

$$S_i(n) = M_i(n)P_{i-1}(n|n)M_i'(n)$$
$$+ N_i(n)P_{i-1}(n-1,n|n)M_i'(n)$$
$$+ M_i(n)P_{i-1}'(n-1, n|n)N_i'(n)$$
$$+ N_i(n)P_{i-1}(n-1|n)N_i'(n) + R_i(n) \tag{3.3}$$

$$K_i(n) = [P_{i-1}(n|n)M_i'(n) + P_{i-1}'(n-1, n|n)N_i'(n)]S_i^{-1}(n) \tag{3.4}$$

$$P_i(n\ n) = P_{i-1}(n|n) - K_i(n)S_i(n)K_i'(n) \tag{3.5}$$

$$\hat{x}_i(n|n) = \hat{x}_{i-1}(n|n) + K_i(n)[y_i(n) - M_i(n)\hat{x}_{i-1}(n|n)$$
$$- N_i(n)\hat{x}_{i-1}(n-1|n)] \tag{3.6}$$

$$W_i(n) = [P_{i-1}(n-1, n|n)\ M_i(n) + P_{i-1}(n-1|n)N_i'(n)]S_i^{-1}(n) \tag{3.7}$$

$$P_i'(n-1,n|n) = P_{i-1}'(n-1,n|n) - K_i(n)S_i(n)W_i'(n) \tag{3.8}$$

$$P_i(n-1|n) = P_{i-1}(n-1|n) - W_i(n)S_i(n)W_i'(n) \tag{3.9}$$

$$\hat{x}_i(n-1|n) = \hat{x}_{i-1}(n-1|n) + W_i(n)[y_i(n)$$
$$- M_i(n)\hat{x}_{i-1}(n|n) - N_i(n)\hat{x}_{i-1}(n-1|n)] \tag{3.10}$$

where

$$P_1'(n-1,n|n) = \phi(n-1)P(n-1|n-1) - K_1(n)S_1(n)W_1'(n) \tag{3.11}$$

$$P_1(n-1|n) = P(n-1|n-1) - W_1(n)S_1(n)W_1'(n) \tag{3.12}$$

$$\hat{x}_1(n-1|n) = \hat{x}(n-1|n-1) + W_1(n)[y_1(n) - M_1(n)\hat{x}(n|n-1) - N_1(n)\hat{x}(n-1|n-1)] \tag{3.13}$$

$$W_1(n) = [P(n-1|n-1)\phi'(n-1)M_1'(n) + P(n-1|n-1)N_1'(n)]S_1^{-1}(n) \tag{3.14}$$

The final estimate $\hat{x}_p(n|n)$ and error covariance $P_p(n|n)$ are the optimal estimate and covariance respectively at time n. It is not necessary to use equations (3.7) through (3.10) on the last iteration.

Now it will be shown that the same results are obtained from both sequential and conventional simultaneous processing for the case p=2 corresponding to a single partition of the measurement equation.

The simultaneous method may be expressed in terms of the partitioned quantities as follows:

For the gain matrix, the result is

$$K(n) = \{P(n|n-1)[M_1' \vdots M_2'] + \phi P(n-1|n-1)[N_1' \vdots N_2']\}.$$

$$\{\begin{bmatrix} M_1 \\ \cdots \\ M_2 \end{bmatrix} P(n|n-1)[M_1' \vdots M_2'] + \begin{bmatrix} N_1 \\ \cdots \\ N_2 \end{bmatrix} P(n-1|n-1)[N_1' \vdots N_2']$$

$$+ \begin{bmatrix} N_1 \\ \cdots \\ N_2 \end{bmatrix} P(n-1|n-1)\phi'[M_1' \vdots M_2'] + \begin{bmatrix} R_1 & \vdots & 0 \\ \cdots & \vdots & \cdots \\ 0 & \vdots & R_2 \end{bmatrix}$$

$$+ \begin{bmatrix} M_1 \\ \cdots \\ M_2 \end{bmatrix} \phi P(n-1|n-1)[N_1' \vdots N_2']\}^{-1} \tag{3.15}$$

$$= [P(n|n-1)M_1' + \phi P(n-1|n-1)N_1' \vdots P(n|n-1)M_2' + \phi P(n-1|n-1)N_2']$$

$$\begin{bmatrix} T_1 & \vdots & T_2 \\ \cdots & \vdots & \cdots \\ T_3 & \vdots & T_4 \end{bmatrix}^{-1} \tag{3.16}$$

where

$$T1 = M_1 P(n|n-1)M_1' + N_1 P(n-1|n-1)N_1' + R_1$$

$$+ N_1 P(n-1|n-1)\phi' M_1' + M_1 \phi P(n-1|n-1)N_1' \tag{3.17}$$

$$T2 = M_1 P(n|n-1)M_2' + N_1 P(n-1|n-1)N_2'$$

$$+ N_1 P(n-1|n-1)\phi' M_2' + M_1 \phi P(n-1|n-1)N_2' \tag{3.18}$$

$$T3 = (T2)' \tag{3.19}$$

$$T4 = M_2P(n|n-1)M_2' + N_2P(n-1|n-1)N_2'$$

$$+ N_2P(n-1|n-1)\phi'M_2' + M_2\phi P(n-1|n-1)N_2' + R_2 \qquad (3.20)$$

For the covariance, the result is

$$P(n|n) = P(n|n-1)-K(n)\{\begin{bmatrix} M_1 \\ \cdots \\ M_2 \end{bmatrix}P(n|n-1) + \begin{bmatrix} N_1 \\ \cdots \\ N_2 \end{bmatrix}P(n-1|n-1)\phi'\}$$

$$= P(n|n-1)-K(n)\begin{bmatrix} M_1P(n|n-1)+N_1P(n-1|n-1)\phi' \\ \cdots \\ M_2P(n|n-1)+N_2P(n-1|n-1)\phi' \end{bmatrix} \qquad (3.21)$$

The estimate in terms of the partitioned quantities is

$$\hat{x}(n|n) = \hat{x}(n|n-1)+K(n)\begin{bmatrix} y_1-M_1\hat{x}(n|n-1)-N_1\hat{x}(n-1|n-1) \\ \cdots \\ y_2-M_2\hat{x}(n|n-1)-N_2\hat{x}(n-1|n-1) \end{bmatrix} \qquad (3.22)$$

The sequential method for the covariance is

$$K_1(n) = [P(n|n-1)M_1'+\phi P(n-1|n-1)N_1']S_1^{-1}(n) \qquad (3.23)$$

$$S_1(n) = M_1P(n|n-1)M_1'+N_1P(n-1|n-1)N_1'$$

$$+ N_1P(n-1|n-1)\phi'M_1'+M_1\phi P(n-1|n-1)N_1'+R_1 \qquad (3.24)$$

$$P_1(n|n) = P(n|n-1)-K_1[M_1P(n|n-1)$$

$$+ N_1P(n-1|n-1)\phi'] \qquad (3.25)$$

$$K_2(n) = [P_1(n|n)M_2'+P_1'(n-1,n|n)N_2']S_2^{-1}(n) \tag{3.26}$$

$$S_2(n) = M_2P_1(n|n)M_2'+N_2P_1(n-1,n|n)M_2'$$

$$+ M_2P_1'(n-1,n|n)N_2'+N_2P_1(n-1|n)N_2' + R_2 \tag{3.27}$$

$$P_2(n|n) = P_1(n|n)-K_2S_2K_2'$$

$$= P(n|n-1)-K_1S_1K_1' \tag{3.28}$$

$$-[P_1(n|n)M_2'+P_1'(n-1,n|n)N_2']S_2^{-1}[M_2P_1(n|n)+N_2P_1(n-1,n|n)] \tag{3.29}$$

Substituting for $P_1(n|n)$, $P_1(n-1,n|n)$ and $K_1(n)$ gives the following factored form

$$P_2(n|n) = P(n|n-1)$$

$$[P(n|n-1)M_1'+\phi P(n-1|n-1)N_1' \vdots P(n|n-1)M_2'+\phi P(n-1|n-1)N_2']$$

$$\begin{bmatrix} T5 & \vdots & T6 \\ \cdots & \cdots & \cdots \\ T7 & \vdots & T8 \end{bmatrix} \begin{bmatrix} M_1P(n|n-1)+N_1P(n-1|n-1)\phi' \\ \cdots\cdots\cdots\cdots \\ M_2P(n|n-1)+N_2P(n-1|n-1)\phi' \end{bmatrix} \tag{3.30}$$

where

$$T5 = S_1^{-1}+\{S_1^{-1}[M_1P(n|n-1)+N_1P(n-1|n-1)\phi']M_2'$$

$$+ W_1'N_2'\}S_2^{-1}\{M_2[P(n|n-1)M_1'+\phi P(n-1|n-1)N_1']S_1^{-1}$$

$$+ N_2W_1\} \tag{3.31}$$

$$T6 = -\{S_1^{-1}[M_1P(n|n-1)+N_1P(n-1|n-1)\phi']M_2' + W_1'N_2'\}S_2^{-1} \tag{3.32}$$

$$T7 = (T6)' \tag{3.33}$$

$$T8 = S_2^{-1} \tag{3.34}$$

Comparing the two results for $P(n|n)$ it is seen that they will be the same if

$$\begin{bmatrix} T1 & \vdots & T2 \\ \cdots & & \cdots \\ T3 & \vdots & T4 \end{bmatrix}^{-1} = \begin{bmatrix} T5 & \vdots & T6 \\ \cdots & & \cdots \\ T7 & \vdots & T8 \end{bmatrix} \tag{3.35}$$

or

$$\begin{bmatrix} T5 & \vdots & T6 \\ \cdots & & \cdots \\ T7 & \vdots & T8 \end{bmatrix} \begin{bmatrix} T1 & \vdots & T2 \\ \cdots & & \cdots \\ T3 & \vdots & T4 \end{bmatrix} = I \tag{3.36}$$

where I is the mxm identity matrix. The conditions then can be expressed as

$$\begin{aligned} T5T1 + T6T3 &= I \\ T7T2 + T8T4 &= I \\ T5T2 + T6T4 &= 0 \\ T7T1 + T8T3 &= 0 \end{aligned} \tag{3.37}$$

where 0 is a null matrix. These conditions are shown to hold by substitution for the indicated terms and then for W_1.

The sequential method for the optimal estimate is

$$\hat{x}_1(n|n) = \hat{x}(n|n-1)+K_1[y_1-M_1\hat{x}(n|n-1)-N_1\hat{x}(n-1|n-1)] \tag{3.38}$$

$$\hat{x}_2(n|n) = \hat{x}_1(n|n)+K_2[y_2-M_2\hat{x}_1(n|n)-N_2\hat{x}_1(n-1|n)] \tag{3.39}$$

Substituting for $\hat{x}_1(n|n)$ and $\hat{x}_1(n-1|n)$ and factoring gives

$$\hat{x}_2(n|n) = [K_1-K_2(M_2K_1+N_2W_1) \vdots K_2] \begin{bmatrix} y_1-M_1\hat{x}(n|n-1)-N_1\hat{x}(n-1|n-1) \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ y_2-M_2\hat{x}(n|n-1)-N_2\hat{x}(n-1|n-1) \end{bmatrix} \tag{3.40}$$

Comparing this with the simultaneous result it can be seen that they are the same if

$$[P(n|n-1)M_1'+\phi P(n-1|n-1)N_1' \vdots P(n|n-1)M_2'+\phi P(n-1|n-1)N_2'] \begin{bmatrix} T1 & \vdots & T2 \\ \cdots & \cdots & \cdots \\ T3 & \vdots & T4 \end{bmatrix}^{-1}$$

$$= [K_1-K_2(M_2K_1+N_2W_1) \vdots K_2] \tag{3.41}$$

Or, using Equation (3.35), the two results will be the same if

$$[P(n|n-1)M_1'+\phi P(n-1|n-1)N_1' \vdots P(n|n-1)M_2'+\phi P(n-1|n-1)N_2'] \begin{bmatrix} T5 & \vdots & T6 \\ \cdots & \cdots & \cdots \\ T7 & \vdots & T8 \end{bmatrix}$$

$$= [K_1-K_2(M_2K_1+N_2W_1) \vdots K_2] \tag{3.42}$$

The term in the 1,2 position is

$$[P(n|n-1)M_1'+\phi P(n-1|n-1)N_1']T6$$

$$+\ [P(n|n-1)M_2'+\phi P(n-1|n-1)N_2']T8$$

Substituting for T6, T8 and then for W_1 and Equations (3.25) and (3.11), the result for K_2 is obtained.

The 1,1 term is

$$[P(n|n-1)M_1'+\phi P(n-1|n-1)N_1']T5$$

$$+\ [P(n|n-1)M_2'+\phi P(n-1|n-1)N_2']T7$$

Substituting for T5, T7 and then for W_1 and using the result just obtained for the 1,2 term, the above expression can be seen to be $[K_1-K_2(M_2K_1+N_2W_1)]$

It has been shown now that the sequential algorithm can be used to process the second component of a partitioned measurement vector. Since the same operations are involved it can be seen that the same algorithm can be used for the third or other components of a partitioned measurement. However since $\hat{x}_2(n-1|n)$, $P_2(n-1|n)$ and $P_2(n-1,n|n)$ are required for any such further partitioned components, it is necessary to show that correct results are obtained sequentially for these quantities.

First consider the sequential solution for $P'(n-1,n|n)$. This is

$$P_1'(n-1,n|n) = \phi P(n-1|n-1) - K_1 S_1 W' \tag{3.43}$$

$$P_2'(n-1,n|n) = P_1'(n-1,n|n) - K_2 S_2 W_2' \tag{3.44}$$

Substituting for K_1, W_1, K_2 and W_2 and factoring leads to

$$P_2'(n-1,n|n) = \phi P(n-1|n-1) -$$

$$[P(n|n-1)M_1' + \phi P(n-1|n-1)N_1' \vdots P(n|n-1)M_2' + \phi P(n-1|n-1)N_2']$$

$$\begin{bmatrix} T5 & \vdots & T6 \\ \cdots & \cdots & \cdots \\ T7 & \vdots & T8 \end{bmatrix} \begin{bmatrix} M_1 \phi P(n-1|n-1) + N_1 P(n-1|n-1) \\ \cdots\cdots\cdots\cdots \\ M_2 \phi P(n-1|n-1) + N_2 P(n-1|n-1) \end{bmatrix} \tag{3.45}$$

The simultaneous solution for $P'(n-1,n|n)$ in terms of the partitioned quantities requires that $W(n)$ be expressed in the same terms. For $W(n)$, the result is

$$W(n) = [P(n-1|n-1)\phi' M' + P(n-1|n-1)N']S^{-1}(n) \tag{3.46}$$

$$= \{P(n-1|n-1)\phi'[M_1' \vdots M_2'] + P(n-1|n-1)[N_1' \vdots N_2']\}S^{-1}(n) \tag{3.47}$$

$$= [P(n-1|n-1)\phi' M_1' + P(n-1|n-1)N_1' \vdots P(n-1|n-1)\phi' M_2' + P(n-1|n-1)N_2']\, S^{-1}(n) \tag{3.48}$$

Now using (3.16), the simultaneous solution for $P'(n-1,n|n)$ is

$$P'(n-1,n|n) = \phi P(n-1|n-1)-K(n)S(n)W'(n) \quad (3.49)$$

$$= \phi P(n-1|n-1)$$

$$-[P(n|n-1)M_1'+\phi P(n-1|n-1)N_1' \vdots P(n|n-1)M_2'+\phi P(n-1|n-1)N_2']$$

$$\begin{bmatrix} T1 & \vdots & T2 \\ \cdots & & \cdots \\ T3 & \vdots & T4 \end{bmatrix}^{-1} \begin{bmatrix} M_1\phi P(n-1|n-1)+N_1P(n-1|n-1) \\ \cdots \\ M_2\phi P(n-1|n-1)+N_2P(n-1|n-1) \end{bmatrix} \quad (3.50)$$

Comparing this equation with (3.45), it can be seen they are the same since the condition for equality already has been obtained as (3.35).

For $P(n-1|n)$ the sequential solution is

$$P_1(n-1|n) = P(n-1|n-1)-W_1(n)S_1(n)W_1'(n) \quad (3.51)$$

$$P_2(n-1|n) = P_1(n-1|n)-W_2(n)S_2(n)W_2'(n) \quad (3.52)$$

Substituting for W_1 and W_2 and factoring the result gives

$$P_2(n-1|n) = P(n-1|n-1) - \quad (3.53)$$

$$[P(n-1|n-1)\phi'M_1'+P(n-1|n-1)N_1' \vdots P(n-1|n-1)\phi'M_2'+P(n-1|n-1)N_2'].$$

$$\begin{bmatrix} T5 & \vdots & T6 \\ \cdots & & \cdots \\ T7 & \vdots & T8 \end{bmatrix} \begin{bmatrix} M_1\phi P(n-1|n-1)+N_1P(n-1|n-1) \\ \cdots \\ M_2\phi P(n-1|n-1)+N_2P(n-1|n-1) \end{bmatrix}$$

The simultaneous solution for $P(n-1|n)$ expressed in terms of the partitioned submatrices of the measurement equation is

$$P(n-1|n) = P(n-1|n-1) = W(n)S(n)W'(n) \tag{3.54}$$

$$= P(n-1|n-1) -$$

$$[P(n-1|n-1)\phi' M_1' + P(n-1|n-1)N_1' \vdots P(n-1|n-1)\phi M_2' + P(n-1|n-1)N_2'] \cdot$$

$$\begin{bmatrix} T1 & \vdots & T2 \\ \cdots & \cdots & \cdots \\ T3 & \vdots & T4 \end{bmatrix}^{-1} \begin{bmatrix} M_1\phi P(n-1|n-1)+N_1P(n-1|n-1) \\ \cdots\cdots\cdots\cdots \\ M_2\phi P(n-1|n-1)+N_2P(n-1|n-1) \end{bmatrix} \tag{3.55}$$

Again comparison of the two solutions shows they are equal because of (3.35).

In terms of partitioned quantities the simultaneous solution for $\hat{x}(n-1|n)$ is

$$\hat{x}(n-1|n) = \hat{x}(n-1|n-1)$$

$$+ W\begin{bmatrix} y_1 - M_1\hat{x}(n|n-1) - N_1\hat{x}(n-1|n-1) \\ \cdots\cdots\cdots\cdots \\ y_2 - M_2\hat{x}(n|n-1) - N_2\hat{x}(n-1|n-1) \end{bmatrix} \tag{3.56}$$

The sequential solution is

$$\hat{x}_1(n-1|n) = \hat{x}(n-1|n-1) + W_1[y_1 - M_1\hat{x}(n|n-1) - N_1\hat{x}(n-1|n-1)] \tag{3.57}$$

$$\hat{x}_2(n-1|n) = \hat{x}_1(n-1|n) + W_2[y_2 - M_2\hat{x}_1(n|n) - N_2\hat{x}_1(n-1|n)] \tag{3.58}$$

Substituting for $\hat{x}_1(n|n)$ and $\hat{x}_1(n-1|n)$ and factoring gives

$$\hat{x}_2(n-1|n) = \hat{x}(n-1|n-1) + [W_1 - W_2(M_2K_1 + N_2W_1) \vdots W_2] \cdot \begin{bmatrix} y_1 - M_1\hat{x}(n|n-1) - N_1\hat{x}(n-1|n-1) \\ \cdots \\ y_2 - M_2\hat{x}(n|n-1) - N_2\hat{x}(n-1|n-1) \end{bmatrix} \tag{3.59}$$

Comparing the two solutions it can be seen that they are the same if

$$[W_1 - W_2(M_2K_1 + N_2W_1) \vdots W_2]$$

$$= [P(n-1|n-1)\phi' M_1' + P(n-1|n-1)N_1' \vdots P(n-1|n-1)\phi' M_2' + P(n-1|n-1)N_2'] \begin{bmatrix} T1 & \vdots & T2 \\ \cdots & \vdots & \cdots \\ T3 & \vdots & T4 \end{bmatrix}^{-1} \tag{3.60}$$

Or, using (3.35), if

$$= [P(n-1|n-1)\phi' M_1' + P(n-1|n-1)N_1' \vdots P(n-1|n-1)\phi' M_2' + P(n-1|n-1)N_2'] \begin{bmatrix} T5 & \vdots & T6 \\ \cdots & \vdots & \cdots \\ T7 & \vdots & T8 \end{bmatrix} \tag{3.61}$$

The term in the 1,2 position is

$$[P(n-1|n-1)\phi' M_1' + P(n-1|n-1)N_1']T6$$

$$+ [P(n-1|n-1)\phi' M_2' + P(n-1|n-1)N_2']T8$$

Substituting for T6, T8 and then for W_1 the desired result for W_2 is obtained.

The 1,1 term is

$$[P(n-1|n-1)\phi'M_1'+P(n-1|n-1)N]T5$$

$$+\ [P(n-1|n-1)\phi'M_2'+P(n-1|n-1N_2']T7$$

Substituting for T5, T7 and then for W_1 and using the result just obtained for the 1,2 term, the above expression can be seen to be $[W_1-W_2(M_2K_1+N_2W_1)]$.

CHAPTER IV. GENERAL COVARIANCE COMPUTATION

The expression for the estimation error covariance $P(n|n)$ given by (2.19) in the delayed state covariance filter is an optimal calculation in that it is valid only when used with the optimal gain matrix. It is desirable to have a completely general equation for $P(n|n)$ that will give the correct error covariance resulting from the use of any arbitrary gain. Such an equation is obtained here in two different forms. The first of the new equations will be more numerically stable since it is the sum of symmetric, positive definite matrices (8) while the second generally will require fewer arithmetic operations. Both forms will be shown to be stabilized against small errors in gain calculation when used with optimal gains.

The results obtained here may be stated as follows:

If a suboptimal gain is used with the delayed state covariance filter, the covariance of the estimation error $P(n|n)$ is given by

$$\begin{aligned} P(n|n) = &\{\phi(n-1)-K(n)[M(n)\phi(n-1)+N(n)]\}P(n-1|n-1)\{\phi(n-1) \\ &- K(n)[M(n)\phi(n-1)+N(n)]\}' + K(n)R(n)K'(n) \\ &+ [I-K(n)M(n)]B(n-1)Q(n-1)B'(n-1)[I-K(n)M(n)]' \end{aligned} \tag{4.1}$$

or

$$P(n|n) = P(n|n-1)-[P(n|n-1)M'(n)$$

$$+ \phi(n-1)P(n-1|n-1)N'(n)]K'(n)$$

$$- K(n)[M(n)P(n|n-1)+N(n)P(n-1|n-1)\phi'(n-1)]$$

$$+ K(n)S(n)K'(n) \tag{4.2}$$

To obtain the first expression for $P(n|n)$, use (2.6) and (2.14)

$$P(n|n) = \mathrm{cov}[x(n)-\hat{x}(n|n)] \tag{4.3}$$

$$= \mathrm{cov}\{\phi(n-1)x(n-1)+B(n-1)u(n-1)-\hat{x}(n|n-1)$$

$$- K(n)[y(n)-M(n)\hat{x}(n|n-1)-N(n)\hat{x}(n-1|n-1)]\} \tag{4.4}$$

Using (2.15) and dropping time notation for convenience

$$P(n|n) = E\{\phi x+Bu-\phi\hat{x}+KM\phi\hat{x}+KN\hat{x}$$

$$-Ky\}\{\phi x+Bu-\phi\hat{x}+KM\phi\hat{x}+KN\hat{x}-Ky\}' \tag{4.5}$$

Using (2.7)

$$P(n|n) = E\{\phi x+Bu-\phi\hat{x}+KM\phi\hat{x}+KN\hat{x}$$

$$-KNx-Kv-KM\phi x-KMBu\}$$

$$\{\phi x+Bu-\phi\hat{x}+KM\phi\hat{x}+KN\hat{x}-KNx-Kv-KM\phi x-KMBu\}' \tag{4.6}$$

$$= E\{(I-KM)\phi(x-\hat{x})-KN(x-\hat{x})$$

$$-Kv+(I-KM)Bu\}\{(I-KM)\phi(x-\hat{x})$$

$$-KN(x-\hat{x})-Kv+(I-KM)Bu\}' \tag{4.7}$$

$$= E(\phi-KM\phi-KN)(x-\hat{x})(x-\hat{x})'(\phi-KM\phi-KN)'$$

$$+EKvv'K'+E(I-KM)Buu'B'(I-KM)'$$

$$-E(\phi-KM\phi-KN)(x-\hat{x})(v'K')$$

$$-E(\phi-KM\phi-KN)(x-\hat{x})(u'B')(I-KM)'$$

$$-E(Kv)(x-\hat{x})'(\phi-KM\phi-KN)'$$

$$+E(Kv)(u'B')(I-KM)'$$

$$+E(I-KM)\ Bu(x-\hat{x})'(\phi-KM\phi-KN)'$$

$$-E(I-KM)Buv'K' \tag{4.8}$$

The last six terms are zero from Comments 1 through 4 and Equation (2.10) so

$$P(n|n) = (\phi-KM\phi-KN)P(n-1|n-1)\ (\phi-KM\phi-KN)'$$

$$+KRK'+(I-KM)BQB'(I-KM)' \tag{4.9}$$

Next, the form for $P(n|n)$ given by Equation (4.2) will be shown to be correct. Expand (4.1) algebraically and rearrange as

$$P(n|n) = \phi P(n-1|n-1)\phi' - \phi P(n-1|n-1)[M\phi+N]'K'$$
$$-K[M\phi+N]P(n-1|n-1)\phi' + KRK'$$
$$+K[M\phi+N]P(n-1|n-1)[M\phi+N]'K'$$
$$+(I-KM)BQB'(I-KM)' \tag{4.10}$$

$$= \phi P(n-1|n-1)\phi' + BQB' + KRK'$$
$$-[\phi P(n-1|n-1)\phi' + BQB']M'K'$$
$$-KM[\phi P(n-1|n-1)\phi' + BQB']$$
$$+KM[\phi P(n-1|n-1)\phi' + BQB']M'K'$$
$$-\phi P(n-1|n-1)N'K' - KNP(n-1|n-1)\phi'$$
$$+K[NP(n-1|n-1)\phi'M'$$
$$+M\phi P(n-1|n-1)N' + NP(n-1|n-1)N']K' \tag{4.11}$$

$$= P(n|n-1) - P(n|n-1)M'K' - KMP(n|n-1)$$
$$+KMP(n|n-1)M'K' + KRK'$$
$$-\phi P(n-1|n-1)N'K' - KNP(n-1|n-1)\phi'$$
$$+K[NP(n-1|n-1)\phi'M' + M\phi P(n-1|n-1)N'$$
$$+NP(n-1|n-1)N']K' \tag{4.12}$$

$$= P(n|n-1) - P(n|n-1)M'K' - KMP(n|n-1)$$
$$+K[MP(n|n-1)M'+R]K'$$
$$-\phi P(n-1|n-1)N'K' - KNP(n-1|n-1)\phi'$$
$$+K[NP(n-1|n-1)\phi'M' + M\phi P(n-1|n-1)N'$$
$$+NP(n-1|n-1)N']K' \tag{4.13}$$

$$= P(n|n-1) - P(n|n-1)M'K' - KMP(n|n-1)$$
$$+KSK' - \phi P(n-1|n-1)N'K' - KNP(n-1|n-1)\phi' \tag{4.14}$$

$$= P(n|n-1) - [P(n|n-1)M' + P(n-1|n-1)N']K'$$
$$-K[MP(n|n-1) + NP(n-1|n-1)\phi'] + KSK' \tag{4.15}$$

which is the same as (4.2). This does not have the property of consisting of the sum of positive definite symmetric matrices as does (4.1), but is more efficient computationally since products of nxn matrices are avoided.

Comment 5: The general expressions for $P(n|n)$, Equations (4.1) and (4.2), are stabilized against first order errors in optimal gain calculation.

This can be demonstrated by incrementing the gain K by ΔK and determining the corresponding ΔP. Ignoring second order terms and using (4.1)

$$P(n|n)+\Delta P(n|n) = [\phi-(K+\Delta K)(M\phi+N)]P(n-1|n-1)$$

$$[\phi-(K+\Delta K)(M\phi+N)]'+(K+\Delta K)R(K+\Delta K)'$$

$$+ [I-(K+\Delta K)M]BQB'[I-(K+\Delta K)M]' \tag{4.16}$$

$$= [\phi-K(M\phi+N)]P(n-1|n-1)[\phi-K(m\phi+N)]'$$

$$-[\Delta K(M\phi+N)]P(n-1|n-1)[\phi-K(M\phi+N)]'$$

$$-[\phi-K(M\phi+N)]P(n-1|n-1)[\Delta K(M\phi+N)]' \tag{4.17}$$

$$+ KRK'+\Delta KRK'+KR(\Delta K)'$$

$$+ (I-KM)BQB'(I-KM)'$$

$$- \Delta KMBQB'(I-KM)'$$

$$- (I-KM)BQB'(\Delta KM)' .$$

The covariance increment is

$$\Delta P(n|n) = \Delta K\{-(M\phi+N)P(n-1|n-1)[\phi-K(M\phi+N)]'$$

$$+RK'-MBQB'(I-KM)'\} + \{\cdot\}'(\Delta K)' \tag{4.18}$$

where the notation $\{\cdot\}$ indicates that both terms in braces in the expression are the same. Thus ΔP consists of two terms which are the transpose of each other.

Rearranging the expression for ΔP gives

$$\Delta P(n|n) = \Delta K\{-(M\phi+N)P(n-1|n-1)\phi'$$

$$- MBQB'+[M\phi+N)P(n-1|n-1)(M\phi+N)'$$

$$+ R+MBQB'M']K'\} + \{\cdot\}'(\Delta K)' \qquad (4.19)$$

$$= \Delta K\{-(M\phi+N)P(n-1|n-1)\phi'-MBQB'$$

$$+ [M\phi P(n-1|n-1)\phi'M'+NP(n-1|n-1)N'$$

$$+ M\phi P(n-1|n-1)N'+NP(n-1|n-1)\phi'M'$$

$$+ R+MBQB'M']K'\}$$

$$+ \{\cdot\}'(\Delta K)' \qquad (4.20)$$

Using Equations (2.16) and (2.18)

$$\Delta P(n|n) = \Delta K\{-(M\phi+N)P(n-1|n-1)\phi'$$

$$- MBQB' + SK'\}$$

$$+ \{\cdot\}'(\Delta K)' \qquad (4.21)$$

Substituting Equations (2.17) and (2.18)

$$\Delta P(n|n) = \Delta K\{-(M\phi+N)P(n-1|n-1)\phi'$$

$$- MBQB'+MP(n|n-1)+NP(n-1|n-1)\phi'\}$$

$$+ \{\cdot\}'(\Delta K)' \qquad (4.22)$$

$$= 0 \qquad (4.23)$$

so the increment in P is zero, at least for first order effects, when using optimal gains.

An identical conclusion can be reached for the second form of $P(n|n)$, Equation (4.2). Proceeding as for (4.1) and ignoring second order terms again

$$
\begin{aligned}
P(n|n)+\Delta P(n|n) &= P(n|n-1) \\
&\quad - [P(n|n-1)M'+\phi P(n-1|n-1)N'][K+\Delta K]' \\
&\quad - [K+\Delta K][MP(n|n-1)+NP(n-1|n-1)\phi'] \\
&\quad + [K+\Delta K]S[K+\Delta K]'
\end{aligned} \tag{4.24}
$$

$$
\begin{aligned}
&= P(n|n-1)-[P(n|n-1)M'+\phi P(n-1|n-1)N']K' \\
&\quad - [P(n|n-1)M'+\phi P(n-1|n-1)N'](\Delta K)' \\
&\quad - K[MP(n|n-1)+NP(n-1|n-1)\phi] \\
&\quad - (\Delta K)[MP(n|n-1)+NP(n-1|n-1)\phi'] \\
&\quad + KSK'+KS\Delta K'+\Delta KSK'
\end{aligned} \tag{4.25}
$$

or

$$
\begin{aligned}
\Delta P(n|n) &= \Delta K[SK'-MP(n|n-1)-NP(n-1|n-1)\phi] \\
&\quad + [KS-P(n|n-1)M'-\phi P(n-1|n-1)N'](\Delta K)'
\end{aligned} \tag{4.26}
$$

$$
= 0 \tag{4.27}
$$

These results should be compared with the result obtained using (2.32). For that case

$$P(n|n) = P(n|n-1)-K[MP(n|n-1)+NP(n-1|n-1)\phi'] \tag{4.28}$$

so

$$P(n|n) + \Delta P(n|n) = P(n|n-1)-[K+\Delta K][MP(n|n-1) + NP(n-1|n-1)\phi'] \tag{4.29}$$

or

$$\Delta P(n|n) = -(\Delta K)[MP(n|n-1)+NP(n-1|n-1)\phi'] \tag{4.30}$$

Thus the conventional equation is not stabilized against errors in gain calculation.

CHAPTER V. DELAYED STATE INFORMATION FILTER

The previous algorithms for the delayed state filter involve propagation of the error covariance from step to step. A new algorithm now will be presented that shows that the delayed state filter also may be implemented in a form in which the inverse of the error covariance matrix is propagated. Estimation algorithms of this type often are referred to as information filters. This technique is very advantageous for the case in which pessimistic values are assigned for $P(0|0)$ (10). For this case the initial covariance would be very large, leading to starting problems if the covariance filter implementation is applied directly. If the inverse of the covariance matrix is used, the starting problems are avoided since $P^{-1}(0|0)$ would be very small. This information filter is equivalent to the covariance form in the sense that the same information is obtained but it is not algebraically identical. Thus the numerical behavior may be substantially different.

For the information filter implementation, the optimal estimate for Problem 2 may be obtained from $d(n|n)$ and $P^{-1}(n|n)$ or $d(n|n-1)$ and $P^{-1}(n|n-1)$ using the definitions

$$d(n|n) = P^{-1}(n|n)\hat{x}(n|n) \tag{5.1}$$

$$d(n|n-1) = P^{-1}(n|n-1)\hat{x}(n|n-1) \tag{5.2}$$

and the following algorithm, if the indicated inverses exist:

$$d(n|n) = d(n|n-1)+C'(n)V^{-1}(n)\{y(n) - N(n)\phi^{-1}(n-1)H(n)d(n|n-1)\} \quad (5.3)$$

$$d(n|n-1) = J(n)\phi^{-T}(n-1)d(n-1|n-1) \quad (5.4)$$

$$C'(n) = M'(n)+J(n)\phi^{-T}(n-1)N'(n) \quad (5.5)$$

$$V(n) = R(n)+N(n)\phi^{-1}(n-1)H(n)J(n)\phi^{-T}(n-1)N'(n) \quad (5.6)$$

$$H(n) = B(n-1)Q(n-1)B'(n-1) \quad (5.7)$$

$$J(n) = I-F(n)B(n-1)[Q^{-1}(n-1)+B'(n-1)F(n)B(n-1)]^{-1}B'(n-1) \quad (5.8)$$

$$F(n) = \phi^{-T}(n-1)P^{-1}(n-1|n-1)\phi^{-1}(n-1) \quad (5.9)$$

$$P^{-1}(n|n-1) = J(n)F(n) \quad (5.10)$$

$$P^{-1}(n|n) = P^{-1}(n|n-1)+C'(n)V^{-1}(n)C(n) \quad (5.11)$$

where

$$\phi^{-T}(n-1) = [\phi^{-1}(n-1)]' \quad (5.12)$$

and I is the nxn identity matrix.

To verify the algorithm, it is necessary to show that both the time update and measurement are correct in the event that a measurement is not available at time n. For $P^{-1}(n|n-1)$ use (2.18) and (5.10) and consider the product

$$P(n|n-1)P^{-1}(n|n-1)$$

$$= [\phi(n-1)\, P(n-1|n-1)\,\phi'(n-1)$$

$$+ B(n-1)Q(n-1)B'(n-1)]J(n)F(n) \tag{5.13}$$

$$= [\phi(n-1)P(n-1|n-1)\phi'(n-1)+B(n-1)Q(n-1)B'(n-1)]\{F(n)$$

$$- F(n)B(n-1)[Q^{-1}(n-1)+B'(n-1)F(n)B(n-1)]^{-1}B'(n-1)F(n)\} \tag{5.14}$$

$$= I+B(n-1)Q(n-1)B'(n-1)F(n)$$

$$- B(n-1)[Q^{-1}(n-1)+B'(n-1)F(n)B(n-1)]^{-1}B'(n-1)F(n)$$

$$- B(n-1)Q(n-1)B'(n-1)F(n)B(n-1)[Q^{-1}(n-1)$$

$$+ B'(n-1)F(n)B(n-1)]^{-1}B'(n-1)F(n) \tag{5.15}$$

$$= I + \{B(n-1)Q(n-1)B'(n-1)$$

$$- [I+B(n-1)Q(n-1)B'(n-1)F(n)]B(n-1)[Q^{-1}(n-1)$$

$$+ B'(n-1)F(n)B(n-1)]^{-1}B'(n-1)\}F(n) \tag{5.16}$$

$$= I + \{B(n-1)Q(n-1)B'(n-1)-B(n-1)Q(n-1)[Q^{-1}(n-1)$$

$$+ B'(n-1)F(n)B(n-1)][Q^{-1}(n-1)$$

$$+ B'(n-1)F(n)B(n-1)]^{-1}B'(n-1)\}F(n) \tag{5.17}$$

$$= I+\{B(n-1)Q(n-1)B'(n-1)$$

$$- B(n-1)Q(n-1)B'(n-1)\}F(n) \tag{5.18}$$

$$= I \ . \tag{5.19}$$

Thus (5.10) is a correct expression for $P^{-1}(n|n-1)$.

To verify Equation (5.4) for the time update of the estimate, start with (2.15) as

$$\hat{x}(n|n-1) = \phi(n-1)\hat{x}(n-1|n-1) \tag{5.20}$$

$$= \phi(n-1)P(n-1|n-1)\phi'(n-1)\phi^{-T}(n-1)P^{-1}(n-1|n-1)\hat{x}(n-1|n-1) \tag{5.21}$$

$P^{-1}(n|n-1)\hat{x}(n|n-1)$

$$=P^{-1}(n|n-1)\phi(n-1)P(n-1|n-1)\phi'(n-1)\phi^{-T}(n-1)P^{-1}(n-1|n-1)\hat{x}(n-1|n-1) \tag{5.22}$$

Using (5.10) and (5.2)

$$d(n|n-1) = J(n)\phi^{-T}(n-1)d(n-1|n-1) \tag{5.23}$$

which is the result stated as Equation (5.4).

Now verify the measurement update portion of the algorithm. For the covariance $P^{-1}(n|n)$ use Equations (5.11) and (2.19) and then substitute from Equations (5.6), (5.5) and (2.17). Consider the product

$$P(n|n)P^{-1}(n|n) = \{P(n|n-1)-K[MP(n|n-1)$$

$$+ NP(n-1|n-1)\phi']\}[P^{-1}(n|n-1)+C'V^{-1}C] \qquad (5.24)$$

$$= \{P(n|n-1)-[P(n|n-1)M'+\phi P(n-1|n-1)N']S^{-1}$$

$$[MP(n|n-1)+NP(n-1|n-1)\phi']\}\{P^{-1}(n|n-1)$$

$$+ [M'+P^{-1}(n|n-1)\phi P(n-1|n-1)N']V^{-1}[M+$$

$$+ NP(n-1|n-1)\phi'P^{-1}(n|n-1)]\} \qquad (5.25)$$

$$= I+P(n|n-1)[M'+P^{-1}(n|n-1)\phi P(n-1|n-1)N']V^{-1}(M+$$

$$+ NP(n-1|n-1)\phi'P^{-1}(n|n-1)]$$

$$- [P(n|n-1)M'+\phi P(n-1|n-1)N']S^{-1}[MP(n|n-1)$$

$$+ NP(n-1|n-1)\phi']P^{-1}(n|n-1)$$

$$- [P(n|n-1)M'+\phi P(n-1|n-1)N']S^{-1}(MP(n|n-1)$$

$$+ NP(n-1|n-1)\phi'][M'+P^{-1}(n|n-1)\phi P(n-1|n-1)N']V^{-1}(M+$$

$$+ NP(n-1|n-1)\phi'P^{-1}(n|n-1)] \qquad (5.26)$$

$$= I+[P(n|n-1)M'+\phi P(n-1|n-1)N']\{V^{-1}-S^{-1}$$

$$- S^{-1}[MP(n|n-1)+NP(n-1|n-1)\phi'][M'+$$

$$+ P^{-1}(n|n-1)\phi P(n-1|n-1)N']V^{-1}\}[M+$$

$$+ NP(n-1|n-1)\phi'P^{-1}(n|n-1)] \tag{5.27}$$

$$= I+[P(n|n-1)M'+\phi P(n-1|n-1)N']\{[I-$$

$$- S^{-1}[MP(n|n-1)+NP(n-1|n-1)\phi'][M'+$$

$$+ P^{-1}(n|n-1)\phi P(n-1|n-1)N']]V^{-1}-S^{-1}\}[M+$$

$$+ NP(n-1|n-1)\phi'P^{-1}(n|n-1)] \tag{5.28}$$

$$= I+[P(n|n-1)M'+\phi P(n-1|n-1)N']\{S^{-1}[S-$$

$$- [MP(n|n-1)+NP(n-1|n-1)\phi'][M'+$$

$$+ P^{-1}(n|n-1)\phi P(n-1|n-1)N']]V^{-1}-S^{-1}\}[M+$$

$$+ NP(n-1|n-1)\phi'P^{-1}(n|n-1)] \tag{5.29}$$

$$= I+[P(n|n-1)M'+\phi P(n-1|n-1)N']\{S^{-1}VV^{-1}-S^{-1}\}$$

$$[M+NP(n-1|n-1)\phi'P^{-1}(n|n-1)] \tag{5.30}$$

$$= I \tag{5.31}$$

Thus the expression given in Equation (5.11) is the inverse of $P(n|n)$.

To obtain Equation (5.3) for the measurement update of the estimate, use Equation (2.14)

$$\hat{x}(n|n) = \hat{x}(n|n-1)+K\{y-M\hat{x}(n|n-1)-N\hat{x}(n-1|n-1) \tag{5.32}$$

$$= [I-KM]\hat{x}(n|n-1)-KN\hat{x}(n-1|n-1)$$

$$+ Ky \tag{5.33}$$

$$P^{-1}(n|n)\hat{x}(n|n) = P^{-1}(n|n)[I-KM]\hat{x}(n|n-1)+P^{-1}(n|n)Ky$$

$$-P^{-1}(n|n)KN\hat{x}(n-1|n-1) \tag{5.34}$$

Using Equations (5.1) and (2.19)

$$d(n|n) = \{P^{-1}(n|n-1)+P^{-1}(n|n)KNP(n-1|n-1)\phi'P^{-1}(n|n-1)\}\hat{x}(n|n-1)$$

$$+ P^{-1}(n|n)Ky$$

$$- P^{-1}(n|n)KNP(n-1|n-1)P^{-1}(n-1|n-1)\hat{x}(n-1|n-1) \tag{5.35}$$

$$= d(n|n-1)+P^{-1}(n|n)Ky$$

$$+ P^{-1}(n|n)KNP(n-1|n-1)\phi'd(n|n-1)$$

$$- P^{-1}(n|n)KNP(n-1|n-1)d(n-1|n-1) \tag{5.36}$$

$$= d(n|n-1)+P^{-1}(n|n)Ky$$

$$- P^{-1}(n|n)KNP(n-1|n-1)[d(n-1|n-1)-\phi' d(n|n-1)] \qquad (5.37)$$

$$= d(n|n-1)+P^{-1}(n|n)Ky$$

$$- P^{-1}(n|n)KNP(n-1|n-1)\phi'[\phi^{-T}d(n-1|n-1)-d(n|n-1)] \qquad (5.38)$$

Using Comment 6 and (5.9)

$$d(n|n) = d(n|n-1)+P^{-1}(n|n)Ky \qquad (5.39)$$

$$- P^{-1}(n|n)KNP(n-1|n-1)\phi' FBQB' d(n|n-1)$$

$$= d(n|n-1) + P^{-1}(n|n)Ky \qquad (5.40)$$

$$- P^{-1}(n|n)KN\phi^{-1}BQB' d(n|n-1)$$

From this equation it can be seen that (5.3) can be obtained if

$$P^{-1}(n|n)K = C'(n)V^{-1}(n) \qquad (5.41)$$

To obtain this result use equations (5.11) and (2.17) and then substitute from (2.16) and Comment 7.

$$P^{-1}(n|n)K(n)$$

$$= [P^{-1}(n|n-1)+C'V^{-1}C][P(n|n-1)M'+\phi P(n-1|n-1)N']S^{-1} \qquad (5.42)$$

$$= \{P^{-1}(n|n-1)[P(n|n-1)M'+\phi P(n-1|n-1)N']$$

$$+ P^{-1}(n|n-1)[P(n|n-1)M'+\phi P(n-1|n-1)N']V^{-1}$$

$$[MP(n|n-1)+NP(n-1|n-1)\phi']P^{-1}(n|n-1)$$

$$[P(n|n-1)M'+\phi P(n-1|n-1)N']\}S^{-1} \tag{5.43}$$

$$= P^{-1}(n|n-1)[P(n|n-1)M'+\phi P(n-1|n-1)N']V^{-1}\{V$$

$$+ [MP(n|n-1)+NP(n-1|n-1)\phi']P^{-1}(n|n-1)$$

$$[P(n|n-1)M'+\phi P(n-1|n-1)N']\}S^{-1} \tag{5.44}$$

Using (5.6) and Comment 8

$$P^{-1}(n|n)K(n)$$

$$= P^{-1}(n|n-1)[P(n|n-1)M'+\phi P(n-1|n-1)N']V^{-1}SS^{-1} \tag{5.45}$$

$$= C'(n)V^{-1}(n) \ . \tag{5.46}$$

Now it can be seen that (5.3) can be obtained from (5.46) and (5.40) which verifies the measurement update of the estimate.

<u>Comment 6</u>: A useful relationship that essentially is a result for $J^{-1}(n)$ is

$$[I+F(n)B(n-1)Q(n-1)B'(n-1)]d(n|n-1)=\phi^{-T}(n-1)d(n-1|n-1) \tag{5.47}$$

Observation of (5.4) and (5.8) indicates this equation can be obtained if

$$J^{-1}(n) = I+F(n)B(n-1)Q(n-1)B'(n-1) \tag{5.48}$$

or if

$$I = [I+F(n)B(n-1)Q(n-1)B'(n-1)]J(n) \tag{5.49}$$

Substituting for $J(n)$, the product is

$$\begin{aligned}
&[I+F(n)B(n-1)Q(n-1)B'(n-1)]\{I-F(n)B(n-1)[Q^{-1}(n-1) \\
&\quad + B'(n-1)F(n)B(n-1)]^{-1}B'(n-1)\} \\
&= I+F(n)B(n-1)Q(n-1)B'(n-1) \\
&\quad - F(n)B(n-1)[Q^{-1}(n-1)+B'(n-1)F(n)B(n-1)]^{-1}B'(n-1) \\
&\quad - F(n)B(n-1)Q(n-1)B'(n-1)F(n)B(n-1)[Q^{-1}(n-1) \\
&\quad + B'(n-1)F(n)B(n-1)]^{-1}B'(n-1)
\end{aligned} \tag{5.50}$$

$$= I+F(n)B(n-1)\{Q(n-1)$$

$$- [Q^{-1}(n-1)+B'(n-1)F(n)B(n-1)]^{-1}$$

$$- Q(n-1)B'(n-1)F(n)B(n-1)[Q^{-1}(n-1)$$

$$+ B'(n-1)F(n)B(n-1)]^{-1}\}B'(n-1) \tag{5.51}$$

$$= I+F(n)B(n-1)\{Q(n-1)[Q^{-1}(n-1)+B'(n-1)F(n)B(n-1)]$$

$$- Q(n-1)B'(n-1)F(n)B(n-1)-I\}[Q^{-1}(n-1)$$

$$+ B'(n-1)F(n)B(n-1)]^{-1}B'(n-1) \tag{5.52}$$

$$= I+F(n)B(n-1)\{I+Q(n-1)B'(n-1)F(n)B(n-1)$$

$$- Q(n-1)B'(n-1)F(n)B(n-1)-I\}[Q^{-1}(n-1)$$

$$+ B'(n-1)F(n)B(n-1)]^{-1}B'(n-1) \tag{5.53}$$

$$= I \tag{5.54}$$

which is the condition indicated in (5.49).

Two relationships that are convenient to use in obtaining the delayed state information filter follow.

Comment 7:

$$P(n|n-1)C'(n) = P(n|n-1)M'(n)+\phi(n-1)P(n-1|n-1)N'(n) \tag{5.55}$$

This can be obtained easily starting with (5.5) and using (5.10).

$$P(n|n-1)C'(n) = P(n|n-1)M'(n)$$

$$+ P(n|n-1)J(n)\phi^{-T}(n-1)N'(n) \tag{5.56}$$

$$= P(n|n-1)M'(n)$$

$$+ P(n|n-1)J(n)\phi^{-T}(n-1)P^{-1}(n-1|n-1)\phi^{-1}(n-1)$$

$$\phi(n-1)P(n-1|n-1)N'(n) \tag{5.57}$$

$$= P(n|n-1)M'(n)$$

$$+ P(n|n-1)J(n)F(n)\phi(n-1)P(n-1|n-1)N'(n) \tag{5.58}$$

$$= P(n|n-1)M'(n)+\phi(n-1)P(n-1|n-1)N'(n) \tag{5.59}$$

which is Equation (5.55).

<u>Comment 8</u>:

$$\phi^{-1}(n-1)H(n-1)J(n)\phi^{-T}(n-1) = P(n-1|n-1)$$

$$- P(n-1|n-1)\phi'(n-1)P^{-1}(n|n-1)\phi(n-1)P(n-1|n-1) \tag{5.60}$$

To obtain this result, first note that

$$J(n) = I-P^{-1}(n|n-1)H(n-1) \tag{5.61}$$

since from Equation (5.10)

$$J^{-1}(n) = F(n)P(n|n-1) \tag{5.62}$$

and using (5.9) and (2.18)

$$F(n)P(n|n-1)[I-P^{-1}(n|n-1)H(n-1)]$$

$$= F(n)[P(n|n-1)-H(n-1)] \tag{5.63}$$

$$= I \tag{5.64}$$

Now premultiply (5.61) by $\phi^{-1}(n-1)H(n-1)$ and postmultiply by $\phi^{-T}(n-1)$. The result, without time notation, is

$$\phi^{-1}HJ\phi^{-T} = \phi^{-1}H\phi^{-T}-\phi^{-1}HP^{-1}(n|n-1)H\phi^{-T} \tag{5.65}$$

$$= \phi^{-1}\{H-HP^{-1}(n|n-1)H\}\phi^{-T} \tag{5.66}$$

$$= \phi^{-1}\{P(n|n-1)-H-[I-HP^{-1}(n|n-1)]P(n|n-1)$$

$$+ [I-HP^{-1}(n|n-1)]H\}\phi^{-T} \tag{5.67}$$

$$= \phi^{-1}\{P(n|n-1)-H-[P(n|n-1)-H]P^{-1}(n|n-1).$$

$$[P(n|n-1)-H]\}\phi^{-T} \tag{5.68}$$

Using Equation (2.18)

$$= \phi^{-1}\{\phi P(n-1|n-1)\phi'$$

$$- \phi P(n-1|n-1)\phi' P^{-1}(n|n-1)\phi P(n-1|n-1)\phi'\}\phi^{-T} \tag{5.69}$$

$$= P(n-1|n-1)-P(n-1|n-1)\phi' P^{-1}(n|n-1)\phi P(n-1|n-1). \tag{5.70}$$

which is the relationship stated as Comment 8.

CHAPTER VI. ARITHMETIC REQUIREMENTS

The computer requirements are an important factor in deciding which of the algorithms to use for a given job. One facet of this data processing problem for the delayed state filter was analyzed in this study.

Tables are presented in this chapter that summarize the arithmetic requirements of the algorithms previously presented and provide a means of comparison. A complete comparison for a specific job would include other variables which cannot be considered in general such as machine logic time (15), available storage (16) and numerical stability. Another factor to be considered is that the specific order in which the matrix multiplications are made to minimize the arithmetic requirements depends on the state dimension n, input dimension r and measurement dimension m. The tables assume n greater than r and much greater than m. Also, in many problems measurements are not available at all times which would require more consideration of the time update portions [$\hat{x}(n|n-1)$ and $P(n|n-1)$] of the algorithms.

The results given for the computations consider symmetry when it exists in that only triangular portions of those matrices are computed. Sparseness is not considered however since that is dependent on the specifics of a given problem.

The arithmetic requirements listed for matrix inverse operations and for solutions of linear algebraic systems assume the use of the Cholesky technique for symmetric matrices (17). According to Fox (18) this technique has significant numerical advantages over other common methods. To solve linear algebraic systems of the form

$$b = Ax \tag{6.1}$$

using the Cholesky method, first solve for the upper triangular matrix R defined such that

$$A = R'R \ . \tag{6.2}$$

Then solve for y where

$$b = R'y \tag{6.3}$$

and then obtain x from

$$y = Rx \tag{6.4}$$

For Cholesky inversion of A, obtain upper triangular R as before such that

$$R'R = A \tag{6.5}$$

Then solve for A^{-1} from

$$RA^{-1} = (R')^{-1} \tag{6.6}$$

In all of the tables Q and R were assumed to be diagonal.

This is nearly always the case for R and can be achieved for Q if necessary by redefining the input matrix B(n). For example for

$$B(n) = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix} \tag{6.7}$$

$$u(n) = [{u_{11} \atop u_{21}}] \tag{6.8}$$

with

$$\text{cov}(u) = \begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix} \tag{6.9}$$

$$q_{12} \neq 0 \tag{6.10}$$

define a new input matrix $\bar{B}(n)$ and a new driving function $\bar{u}(n)$ as

$$\bar{B}(n) = \begin{bmatrix} b_{11} & (\frac{q_{12}}{q_{22}} b_{11} + b_{12}) \\ b_{21} & (\frac{q_{12}}{q_{22}} b_{21} + b_{22}) \\ b_{31} & (\frac{q_{12}}{q_{22}} b_{31} + b_{32}) \end{bmatrix} \tag{6.11}$$

$$\bar{u}(n) = \begin{bmatrix} (u_{11} - \frac{q_{12}}{q_{22}} u_{21}) \\ \\ u_{21} \end{bmatrix} \tag{6.12}$$

It is easily seen that

$$B(n)u(n) = \bar{B}(n)\bar{u}(n) \tag{6.13}$$

and

$$\text{cov}(\bar{u}) = \begin{bmatrix} (q_{11} - \frac{q_{12}^{2}}{q_{22}}) & 0 \\ \\ 0 & q_{22} \end{bmatrix} \tag{6.14}$$

It also should be pointed out that for complete arithmetic analysis of the information algorithm it is necessary to know the portion of times for which the state estimate is required. More computations are necessary if $\hat{x}$ is required at all times or if the error covariance is desired.

The tables give the number of multiplications plus divisions, the number of additions plus subtractions and the number of square roots required for each individual operation of each algorithm. Totals are given separately for time and measurement update portions of the algorithms.

As an example of the use of the tables, the arithmetic requirements of the nav-sat problem studied by Winger (19) were determined and the results listed in Table 6. The

Table 1. Conventional covariance filter

Operation	x, ÷	+,-	$\sqrt{\ }$
$(\phi)(P)$	n^3	n^3-n^2	
$(\phi P)(\phi')$	$\frac{1}{2}n^3 + \frac{1}{2}n^2$	$\frac{1}{2}n^3-\frac{1}{2}n$	
$(B)(Q)$	nr		
$(BQ)(B')$	$\frac{1}{2}n^2r + \frac{1}{2}nr$	$\frac{1}{2}n^2r + \frac{1}{2}nr - \frac{1}{2}n^2 - \frac{1}{2}n$	
$(\phi P\phi')+(BQB')$		$\frac{1}{2}n^2 + \frac{1}{2}n$	
$(\phi)(\hat{x})$	n^2	n^2-n	
Time update	$\frac{3}{2}n^3 + \frac{3}{2}n^2 + \frac{3}{2}nr + \frac{1}{2}n^2r$	$\frac{3}{2}n^3 - \frac{3}{2}n + \frac{1}{2}n^2r + \frac{1}{2}nr$	0
$(P)(M')$	n^2m	n^2m-nm	
$(M)(PM')$	$\frac{1}{2}nm^2 + \frac{1}{2}nm$	$\frac{1}{2}nm^2 + \frac{1}{2}nm - \frac{1}{2}m^2 - \frac{1}{2}m$	
$(P)(N')$	n^2m	n^2m-nm	

Table 1 (Continued)

Operation	x, ÷	+,-	$\sqrt{\ }$
(N)(PN')	$\frac{1}{2}nm^2 + \frac{1}{2}nm$	$\frac{1}{2}nm^2 + \frac{1}{2}nm - \frac{1}{2}m^2 - \frac{1}{2}m$	
(ϕ)(PN')	n^2m	n^2m-nm	
(M)(ϕPN')	nm^2	nm^2-m^2	
( )+( )+( )+( )		$3m^2+m$	
(PM')+(ϕPN')		nm	
$(S)^{-1}$	$\frac{1}{2}m^3 + m^2 - \frac{1}{2}m$	$\frac{1}{2}m^3-\frac{1}{2}m^2$	m
[PM'+](S^{-1})	nm^2	nm^2-nm	
[PM'+](K')	$\frac{1}{2}n^2m + \frac{1}{2}nm$	$\frac{1}{2}n^2m + \frac{1}{2}nm - \frac{1}{2}n^2 - \frac{1}{2}n$	
(P)-( )		$\frac{1}{2}n^2 + \frac{1}{2}n$	

Table 1 (Continued)

Operation	x, ÷	+,-	$\sqrt{\ }$
$(M)(\hat{x})$	nm	nm-m	
$(N)(\hat{x})$	nm	nm-m	
$(y)-(\ )-(\ )$		2m	
$(K)(y-)$	nm	nm-n	
$(\hat{x})+(\ )$		n	
measurement update	$\frac{7}{2}n^2m + 3nm^2 + \frac{9}{2}nm + \frac{1}{2}m^3 + m^2 - \frac{1}{2}m$	$\frac{7}{2}n^2m + 3nm^2 + \frac{3}{2}nm + \frac{1}{2}m^3 + \frac{1}{2}m^2$	m

Table 2. Sequential measurement processing (scalar case)

Operation	x, ÷	+, -	$\sqrt{\ }$
PM'	n^2	n^2-n	
M(PM')	n	n-1	
PN'	n^2	n^2-n	
N(PN')	n	n-1	
P'N'	n^2	n^2-n	
M(P'N')	n	n-1	
S		4	
K	n	n	
P(n|n)	$\frac{1}{2}n^2 + \frac{1}{2}n$	$\frac{1}{2}n^2 + \frac{1}{2}n$	
PM'	n^2	n^2-n	
W	n	n	
P(n-1,n|n)	n^2	n^2	
P(n-1|n)	$\frac{1}{2}n^2 + \frac{1}{2}n$	$\frac{1}{2}n^2 + \frac{1}{2}n$	
$M\hat{x}$	n	n-1	
$N\hat{x}$	n	n-1	
$\hat{x}$	n	n+2	

Table 2 (Continued)

Operation	x, ÷	+, -	$\sqrt{}$
$\hat{x}$	n	n	
Measurement update	$6n^2 + 10n$	$6n^2 + 6n - 1$	0

Table 3. General covariance computation (Form I)

Operation	x, ÷	+, -	$\sqrt{\ }$
$(M)(\phi)$	n^2m	$n^2m - nm$	
$(M\phi)+(N)$		nm	
$(K)(M\phi+N)$	n^2m	n^2m-n^2	
$(\phi)-(\ )(\ )$		n^2	
$[\phi-(\ )(\ )]P$	n^3	n^3-n^2	
$\{\ \}[\ ]'$	$\frac{1}{2}n^3 + \frac{1}{2}n^2$	$\frac{1}{2}n^3 - \frac{1}{2}n$	
$(K)(R)$	nm		
$(KR)(K')$	$\frac{1}{2}n^2m + \frac{1}{2}nm$	$\frac{1}{2}n^2m + \frac{1}{2}nm - \frac{1}{2}n^2 - \frac{1}{2}n$	
$(K)(M)$	n^2m	n^2m-n^2	
$I-(KM)$		n	
$(I-KM)(B)$	n^2r	n^2r-nr	
$(I-KM)(B)(Q)$	nr		
$[I-KM][(I-KM)(B)]'$	$\frac{1}{2}n^2r + \frac{1}{2}nr$	$\frac{1}{2}n^2r + \frac{1}{2}nr - \frac{1}{2}n^2 - \frac{1}{2}n$	
Total	$\frac{3}{2}n^3 + \frac{1}{2}n^2 + \frac{7}{2}n^2m$	$\frac{3}{2}n^3 - 3n^2 - \frac{1}{2}n + \frac{7}{2}n^2m$	0
	$+ \frac{3}{2}nm + \frac{3}{2}n^2r + \frac{3}{2}nr$	$+ \frac{1}{2}nm + \frac{3}{2}n^2r - \frac{1}{2}nr$	

Table 4. General covariance computation (Form II)

Operation	$\times, \div$	$+, -$	$\sqrt{\ }$
[PM'+ϕPN'](K')	$\frac{1}{2}n^2m + \frac{1}{2}nm$	$\frac{1}{2}n^2m + \frac{1}{2}nm - \frac{1}{2}n^2 - \frac{1}{2}n$	
(K) (S)	nm^2	nm^2-nm	
(KS)(K')	n^2m	$n^2m - n^2$	
P		$\frac{3}{2}n^2 + \frac{3}{2}n$	
Total	$\frac{3}{2}n^2m + nm^2 + \frac{1}{2}nm$	$\frac{3}{2}n^2m + nm^2 - \frac{1}{2}nm + n$	0

Table 5. Information filter

Operation	x, ÷	+, -	$\sqrt{\ }$
$\phi^{-T}P^{-1}$	n^3	n^3-n^2	
$(\phi^{-T}P^{-1})(\phi^{-1})$	$\frac{1}{2}n^3 + \frac{1}{2}n^2$	$\frac{1}{2}n^3 - \frac{1}{2}n$	
FB	n^2r	n^2r-nr	
(B')(FB)	$\frac{1}{2}nr^2 + \frac{1}{2}nr$	$\frac{1}{2}nr^2 + \frac{1}{2}nr - \frac{1}{2}r^2 - \frac{1}{2}r$	
Q^{-1}	r		
$Q^{-1}+B'FB$		r	
$[Q^{-1}+B'FB)^{-1}$	$\frac{1}{2}r^3 + r^2 - \frac{1}{2}r$	$\frac{1}{2}r^3 - \frac{1}{2}r^2$	
$(FB)(Q^{-1}+B'FB)^{-1}$	nr^2	nr^2-nr	
$[FB(\ )^{-1}](FB)'$	$\frac{1}{2}n^2r + \frac{1}{2}nr$	$\frac{1}{2}n^2r + \frac{1}{2}nr - \frac{1}{2}n^2 - \frac{1}{2}n$	
$F-[\ ](FB)'$		$\frac{1}{2}n^2 + \frac{1}{2}n$	
$\phi^{-T}d$	n^2	n^2-n	
$(B')(\phi^{-T}d)$	nr	$nr-r$	
$[FB(\ )^{-1}][B'\phi^{-T}d]$	nr	$nr-n$	

Table 5 (Continued)

Operation	x, ÷	+, -	$\sqrt{\ }$
time update	$\frac{3}{2}[n^3+n^2+n^2r+nr^2+2nr+\frac{1}{3}r+\frac{1}{3}r^3+\frac{2}{3}r^2]$	$\frac{1}{2}[3n^3-5n+3n^2r+2nr+3nr^2+r^3-2r^2-r]$	r
$\phi^{-T}N'$	n^2m	n^2m-nm	
$(B')(\phi^{-T}N')$	nmr	$nmr-mr$	
$(Q^{-1}+B'FB)^{-1}(B'\phi^{-T}N')$	mr^2	$mr^2 - mr$	
$[B][(\ )^{-1}(\ )]$	nmr	$nmr - nm$	
$[F][B(\ )^{-1}(\ )]$	n^2m	$n^2m - nm$	
C'		$2nm$	
$[N\phi^{-1}][B(\ )^{-1}(\ )]$	nm^2	$nm^2 - m^2$	
V^{-1}	$\frac{1}{2}m^3 + m^2 - \frac{1}{2}m$	$\frac{1}{2}m^3 + m - \frac{1}{2}m^2$	m
$C'V^{-1}$	nm^2	$nm^2 - nm$	
$(C'V^{-1})(C)$	$\frac{1}{2}n^2m + \frac{1}{2}nm$	$\frac{1}{2}n^2m + \frac{1}{2}nm - \frac{1}{2}n^2 - \frac{1}{2}n$	
P^{-1}		$\frac{1}{2}n^2 + \frac{1}{2}n$	
$B'd$	nr	$nr-r$	
$(Q)(B'd)$	r		

Table 5 (Continued)

Operation	x, ÷	+, -	$\sqrt{\ }$
$[N\phi^{-1}B][QB'd]$	mr	$mr - m$	
$(C'V^{-1})(y-)$	nm	$nm - n + m$	
d		n	
measurement update	$\frac{5}{2}n^2m+2nmr-mr^2+2nm^2+\frac{3}{2}nm$ $+nr+r+mr+\frac{1}{2}m^3+m^2-\frac{1}{2}m$	$\frac{5}{2}n^2m-\frac{1}{2}nm+2nmr-mr$ $+mr^2-m^2+\frac{1}{2}m^3+m-\frac{1}{2}m^2$ $+ 2nm^2+nr-r$	m
solve for $\hat{x}(n\|n)$			
$d(n\|n)$ $=P^{-1}(n\|n)\hat{x}(n\|n)$	$\frac{1}{6}n^3 + \frac{3}{2}n^2 + \frac{1}{3}n$	$\frac{1}{6}n^3 + n^2 - \frac{7}{6}n$	n

parameters for this problem are $n=16$, $r=9$ and $m=2$.

With respect to this specific problem only, it can be seen that the penalty for sequential processing is reasonably small considering the advantage of determining the effects of each measurement component separately. Also it can be seen that there is a substantial penalty for using Form I of the general covariance calculation. The information filter does not appear attractive but could be used if starting problems exist. After the effect of the initial conditions has diminished, the problem could be converted to one of the covariance forms by inverting the inverse error covariance matrix.

Table 6. Arithmetic requirements of nav-sat problem

Filter	Update	x,÷	+,-	$\sqrt{\ }$
covariance	time	7896	7344	0
simultaneous	measurement	2135	2038	2
covariance	time	7896	7344	0
sequential	measurement	2713	2600	0
covariance	time	7896	7344	0
general (Form I)	measurement	13647	12326	2
covariance	time	7896	7344	0
general (Form II)	measurement	2711	2598	2
information	time	12810	11927	9
information	measurement	3444	3167	18

CHAPTER VII. SUMMARY

Algorithms are presented which give optimal estimates of the state of a discrete-time system for the case in which the measurements are a linear function of both the state at the time of the measurement and of the state at the preceding instance of time. These new algorithms are applicable to a larger class of conditions or provide estimates in a more desirable manner than those previously existing.

The sequential measurement processing algorithm allows components of a vector measurement to be processed individual ly so the effects of each may be examined independently. Previous results were limited to simultaneous processing of all components of the measurement. This technique also has significant computational advantages since matrix inverse operations are performed on smaller matrices or on scalars.

The general covariance algorithm provides a means of determining the correct error covariance resulting from the use of any arbitrary gain matrix, thus admitting the possibility of suboptimal techniques designed to reduce computational effort. Results obtained by previous authors were limited to the case of optimal gains. Two forms of the result are given with first being more numerically stable than either the second or the previous algorithm if used with optimal gains. The second form generally requires fewer arithmetic operations than the first but more than the previous algorithm if used with optimal gains. Both forms are shown to be stabilized against small errors in gain calculation when used with optimal gains.

An information algorithm for the delayed state filtering problem also is developed. This algorithm is advantageous when very pessimistic values are assigned for initial error covariance, since the usual covariance algorithm has starting problems for this circumstance. This new algorithm is equivalent to, but not algebraically identical with, the conventional algorithm thus allowing the possibility of superior numerical performance.

A comparison of the arithmetic computational requirements of all the new algorithms as well as the conventional algorithm is presented so one of the major factors in selecting an algorithm for a particular application easily can be determined.

LITERATURE CITED

1. Sorenson, H. W. and Stubberud, A. R. "Linear Estimation Theory." Theory and Applications of Kalman Filtering. Edited by C. T. Leondes. North Atlantic Treaty Organization AGARDograph 139, 1970.

2. Wiener, Norbert. Extrapolation, Interpolation, and Smoothing of Stationary Time Series. New York: J. Wiley, 1949.

3. Kalman, R. E. "A New Approach to Linear Filtering and Prediction Problems." Journal of Basic Engineering, 82 (March, 1960), 35-45.

4. Sorenson, H. W. "Kalman Filtering Techniques." Advances in Control Systems. Volume 3. Edited by C. T. Leondes. New York: Academic Press, 1966.

5. Nahi, N. E. Estimation Theory and Applications. New York: J. Wiley, 1969.

6. Meditch, J. S. Stochastic Optimal Linear Estimation and Control. New York: McGraw-Hill, 1969.

7. Brown, R. G. and Hartman, G. L. "Kalman Filter with Delayed States as Observables." Proceedings of the National Electronics Conference, 24 (1968), 67-72.

8. Jazwinski, A. H. Stochastic Processes and Filtering Theory. New York: Academic Press, 1969.

9. Parzen, E. Modern Probability Theory and Its Applications. New York: J. Wiley, 1960.

10. Kaminski, Paul G.; Bryson, Arthur E., Jr.; and Schmidt, Stanley F. "Discrete Square Root Filtering: A Survey of Current Techniques." IEEE Transactions on Automatic Control, AC-16 (December, 1971), 727-736.

11. Brown, R. G. and Hagerman, L. L. "An Optimum Inertial/Dopper-Satellite Navigation System." Journal of the Institute of Navigation, 16 (Fall, 1969), 260-269.

12. Stansell, Thomas A., Jr. "The Navy Navigation Satellite System: Description and Status." Journal of the Institute of Navigation, 15 (Fall, 1968), 229-243.

13. Stuva, R. L. "Computational Aspects of Kalman Filtering with Delayed States as Observables." Unpublished M.S. thesis, Iowa State University, 1973.

14. Brown, R. G. Private Communication. Ames, Iowa. 1973.

15. Mendel, Jerry M. "Computational Requirements for a Discrete Kalman Filter." IEEE Transactions on Automatic Control, AC-16 (December, 1970), 748-758.

16. Gura, I. A. and Bierman, A. B. "On Computational Efficiency of Linear Filtering Algorithms." Automatica, 7 (May, 1971), 299-314.

17. Faddeeva, V. N. Computational Methods of Linear Algebra. Translated from Russian by C. D. Benster. New York: Dover Publications, 1959.

18. Fox, Leslie. An Introduction to Numerical Linear Algebra. Oxford: Clarendon Press, 1964.

19. Winger, D. J. "Error Analysis of an Integrated Inertial/Doppler-Satellite Navigation System with Continuous and Multiple Satellite Coverage." Unpublished Ph.D. dissertation, Iowa State University, 1971.

ACKNOWLEDGMENTS

The author wishes to thank the chairman of his committee, Professor R. G. Brown, and the other members who provided invaluable assistance and encouragement at critical times. He also wishes to thank the many forgotten taxpayers who provided financial support at various times in the form of a fellowship and Veterans Educational Assistance without which this work would not have been completed. The author acknowledges research support from Office of Naval Research contract N00014-68A-0162.

The author is indebted to the members of his family whose support was always unequivocal.